2.2 GROWTH OF FUNCTIONS

DEF: Let f and g be functions $\mathcal{R} \to \mathcal{R}$. Then f is ***asymptotically dominated*** by g if

$$(\exists K \in \mathcal{R})(\forall x > K)[f(x) \leq g(x)]$$

NOTATION: $f \preceq g$.

Remark: This means that eventually, there is an location $x = K$, after which the graph of the function g lies above the graph of the function f.

BIG OH CLASSES

DEF: Let f and g be functions $\mathcal{R} \to \mathcal{R}$. Then f is ***in the class*** $\mathcal{O}(g)$ ("***big-oh of g***") if

$$(\exists C \in \mathcal{R})[f \preceq Cg]$$

NOTATION: $f \in \mathcal{O}(g)$.

DISAMBIGUATION: Properly understood, $\mathcal{O}(g)$ is the class of all functions that are asymptotically dominated by any multiple of g.

TERMINOLOGY NOTE: The phrase "f is big-oh of g" makes sense if one imagines either that the word "in" preceded the word "big-oh", or that "big-oh of g" is an adjective.

Example 2.2.1: $4n^2 + 21n + 100 \in \mathcal{O}(n^2)$
Proof: First suppose that $n \geq 0$. Then

$$\begin{aligned} 4n^2 + 21n + 100 &\leq 4n^2 + 24n + 100 \\ &\leq 4(n^2 + 6n + 25) \\ &\leq 8n^2 \text{ which holds whenever} \end{aligned}$$

$n^2 \geq 6n + 25$, which holds whenever
$n^2 - 6n + 9 \geq 34$, which holds whenever
$n - 3 \geq \sqrt{34}$, which holds whenever
$n \geq 9$. Thus,

$$(\forall n \geq 9)[4n^2 + 21n + 100 \leq 8n^2].$$

Remark: We notice that n^2 itself is asymptotically dominated by $4n^2 + 21n + 100$. However, we proved that $4n^2 + 21n + 100$ is asymptotically dominated by $8n^2$, a multiple of n^2.

WITNESSES

This operational definition of membership in a big-oh class makes the definition of asymptotic dominance explicit.

DEF: Let f and g be functions $\mathcal{R} \to \mathcal{R}$. Then f is ***in the class*** $\mathcal{O}(g)$ (“***big-oh of g***”) if

$$(\exists C \in \mathcal{R})(\exists K \in \mathcal{R})(\forall x > K)[Cg(x) \geq f(x)]$$

DEF: In the definition above, a multiplier C and a location K on the x-axis after which $Cg(x)$ dominates $f(x)$ are called the ***witnesses*** to the relationship $f \in \mathcal{O}(g)$.

Example 2.2.1, continued: The values $C = 8$ and $M = 9$ are witnesses to the relationship

$$4n^2 + 21n + 100 \in \mathcal{O}(n^2).$$

Larger values of C and K could also serve as witnesses. However, a value of C less than or equal to 4 could not be a witness.

CLASSROOM EXERCISE

If one chooses the witness $C = 5$, then $K = 30$ could be a co-witness, but $K = 9$ could not.

Lemma 2.2.1. $(x+1)^n \in \mathcal{O}(x^n)$.

Proof: Let C be the largest coefficient in the (binomial) expansion of $(x+1)^n$, which has $n+1$ terms. Then $(x+1)^n \leq C(n+1)x^n$. $\diamondsuit$

Example 2.2.2: The proof of Lemma 2.2.1 uses the witnesses

$$C = \binom{n}{\lfloor \frac{n}{2} \rfloor} \text{ and } K = 0$$

Theorem 2.2.2. *Let $p(x)$ be a polynomial of degree n. Then $p(x) \in \mathcal{O}(x^n)$.*

Proof: Informally, just generalize Example 2.2.1. Formally, just apply Lemma 2.2.1. $\diamondsuit$

Example 2.2.3: $100n^5 \in \mathcal{O}(e^n)$. Observing that $n = e^{\ln n}$ inspires what follows.

Proof: Taking the upper Riemann sum with unit-sized intervals for $\ln x = \int_1^n \frac{dx}{x}$ implies for $n > 1$ that

$$\begin{aligned}
\ln(n) &< \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \\
&\leq \left(\frac{1}{1} + \cdots + \frac{1}{5}\right) + \frac{1}{6} + \cdots + \frac{1}{n} \\
&\leq \left(\frac{1}{1} + \cdots + \frac{1}{5}\right) + \frac{1}{6} + \cdots + \frac{1}{6} \\
&\leq 5 + \frac{n-5}{6}
\end{aligned}$$

Therefore, $6 \ln n \leq n + 25$, and accordingly,

$100n^5 = 100 \cdot e^{5 \ln n} < 100 \cdot e^{n+25} < e^{32} \cdot e^n$ $\diamondsuit$

We have used the witnesses $C = e^{32}$ and $K = 0$.

Theorem 2.2.3. *Powers dominate logs.*

Proof: See Example 2.2.3. $\diamondsuit$

Theorem 2.2.4. *Exponentials dominate polynomials.*

Proof: See Example 2.2.3. $\diamondsuit$

Example 2.2.4: $2^n \in \mathcal{O}(n!)$.
Proof:

$$\begin{aligned}\overbrace{2 \cdot 2 \cdots 2}^{n \text{ times}} &= 2 \cdot 1 \cdot \overbrace{2 \cdot 2 \cdots 2}^{n-1 \text{ times}} \\ &\le 2 \cdot 1 \cdot 2 \cdot 3 \cdots n = 2n!\end{aligned}$$

We have used the witnesses $C = 2$ and $K = 0$.

BIG-THETA CLASSES

DEF: Let f and g be functions $\mathcal{R} \to \mathcal{R}$. Then f is ***in the class*** $\Theta(g)$ ("***big-theta of g***") if $f \in \mathcal{O}(g)$ and $g \in \mathcal{O}(f)$.