Language Operations

Language Operations

- Operations that can be used to construct languages from other languages
- Since languages are sets, we can use set operations:
  - Union,
  - Intersection
  - Complement
  - Set difference
- Additional operations that strictly deal with strings
  - Concatenation
  - Star
- Example
  - A = {good, bad}
  - B = {boy, girl}

$A \cup B = \{\texttt{good}, \texttt{bad}, \texttt{boy}, \texttt{girl}\}$,

$A \circ B = \{\texttt{goodboy}, \texttt{goodgirl}, \texttt{badboy}, \texttt{badgirl}\}$, and

$A^* = \{\varepsilon, \text{good}, \text{bad}, \text{goodgood}, \text{goodbad}, \text{badgood}, \text{badbad}, \text{goodgoodgood}, \text{goodgoodbad}, \text{goodbadgood}, \text{goodbadbad}, \ldots\}$.

Closure of Regular Languages (FA-recognizable)

- The set of FA-recognizable languages is **closed** under all six string operations.
  - If we start with regular languages and apply the operations, a new regular language is created.
    - May not work with previous finite automata but will for some finite automata

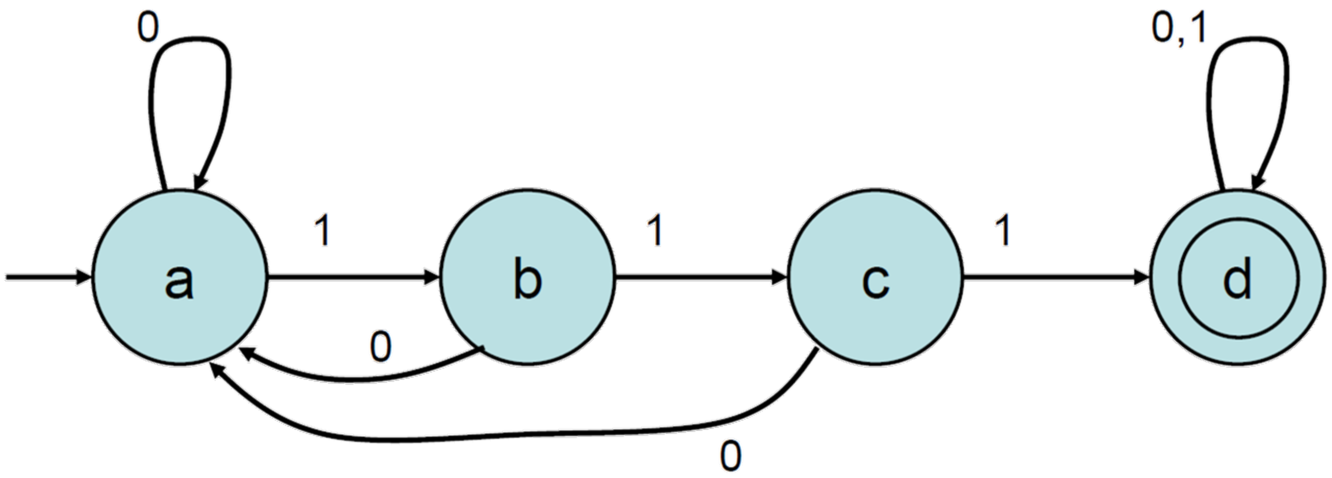
- Theorem 1:
  - FA-recognizable languages are closed under complement

- Proof:
  - Start with a language L_1 over alphabet Σ, recognized by some FA, M_1
  - Produce another FA, M_2, with $L(M_2) = \Sigma^* - L(M_1)$.
    - Just interchange accepting and non-accepting states
  - The new language is recognized by a finite automata and is considered FA-recognizable

Complement of Example 1

- Theorem 1: FA-recognizable languages are closed under complement
- Proof: Interchange accepting and non-accepting states

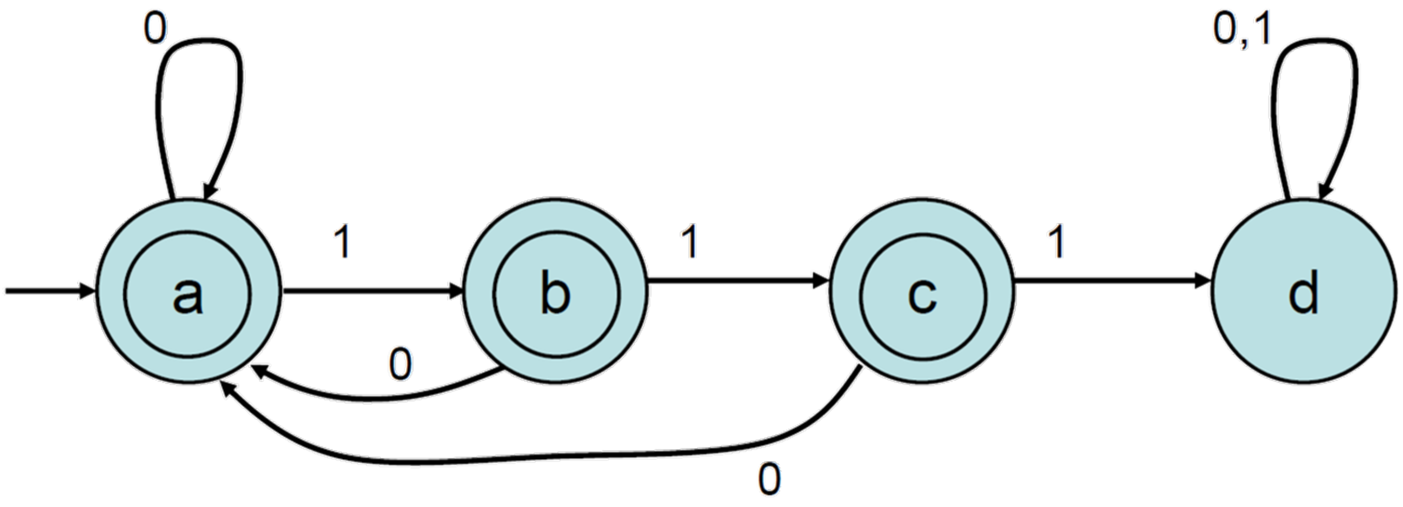
- Example: FA for { w | w does not contain 111 }
  - Start with FA for { w | w contains 111 }:



- Only accepted strings with 111 substring
- Convert to complement language

Complement of Example 1

- Example: FA for { w | w does not contain 111 }
  - Interchange accepting and non-accepting states



- States a,b, and c become accept states
- State d becomes a non-accept state
- Only way to reach d is to have a string with a 111 substring.
  - New FA only recognizes strings that do not have a 111 substring

Closure under Intersection

- Theorem 2: FA-Recognizable languages are closed under **intersection**

- Proof
  - Start with FAs M_1 and M_2 for the same alphabet Σ
  - Get another FA, M_3, with $L(M_3) = L(M_1) \cap L(M_2)$

- Reasoning
  - Run M_1 and M_2 "in parallel" on the same input
    - If both reach accepting states, accept

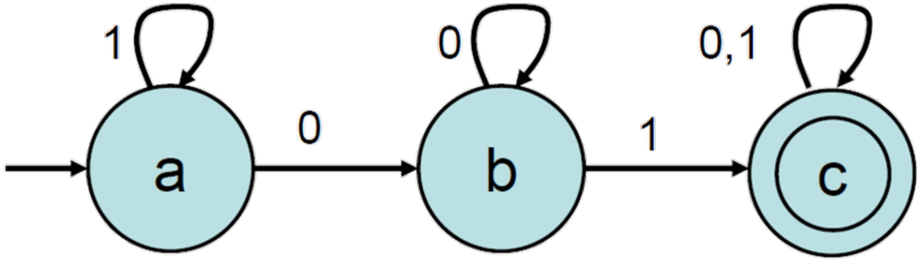
- Example
  - $L(M_1)$: Contains substring 01
  - $L(M_2)$: Contains an odd number of ones
  - $L(M_3)$: Contains 01 and has an odd number of 1s

Closure under Intersection

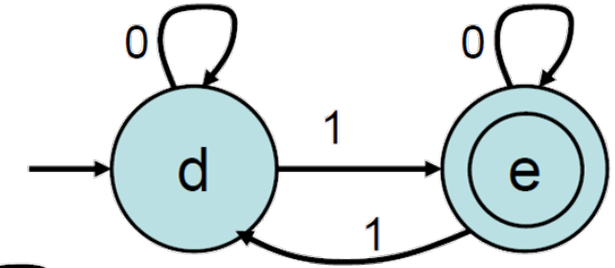
- Run both "in parallel"
  - Only accept string if both accept
- Symbols combine to become new states
  - $\Sigma_1 = \{a,b,c\}$
  - $\Sigma_2 = \{d,e\}$
  - $\Sigma_3 = \{ad,ae,bd,be,cd,ce\}$

- Example:

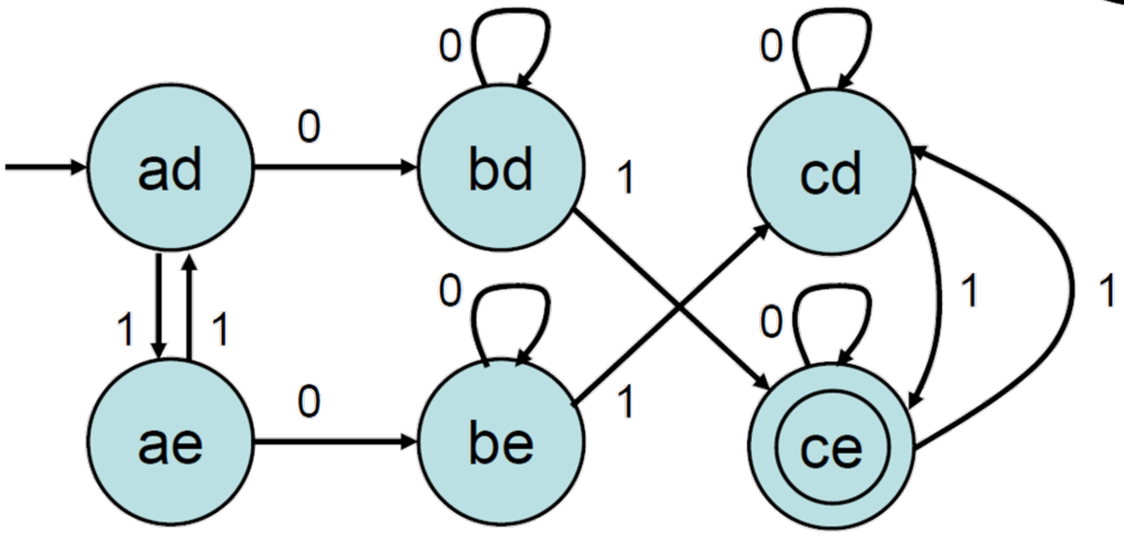
M_1: Substring 01



M_2: Odd number of 1s



M_3:



Closure under Intersection

- New Formal Definition
  - $M_1 = (Q_1,\Sigma_1,\delta_1,q_{01},F_1)$
  - $M_2 = (Q_2,\Sigma_2,\delta_2,q_{02},F_2)$

- Define $M_3 = (Q_3,\Sigma_3,\delta_3,q_{03},F_3)$, where
  - $Q_3 = Q_1 \times Q_2$
    - Cartesian Product, $\{ (q_1,q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \}\}$
  - $\Sigma_3 = \{0,1\}$
  - $\delta_3((q_1,q_2),a) = (\delta_1(q_1,a), \delta_2(q_2,a))$
  - $q_{03} = (q_{01},q_{02})$
  - $F_3 = F_1 \times F_2 = \{ (q_1,q_2) \mid q_1 \in F_1 \text{ and } q_2 \in F_2 \}\}$

Closure under Union

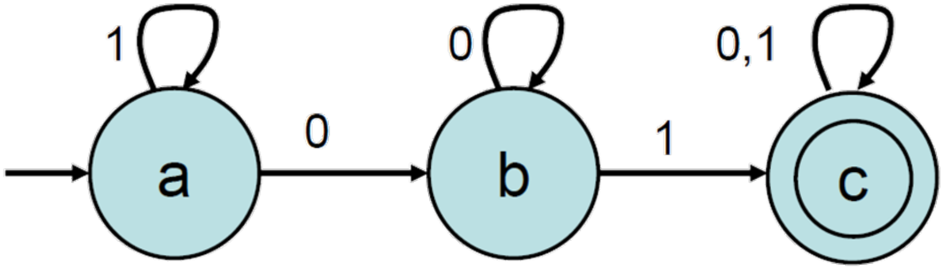
- Theorem 3: FA-Recognizable languages are closed under **union**
- Proof
  - Similar to intersection
  - Start with FAs M_1 and M_2 for the same alphabet Σ
  - Get another FA, M_3, with $L(M_3) = L(M_1) \cup L(M_2)$
- Reasoning
  - Run M_1 and M_2 "in parallel" on the same input
    - If either reach accepting states, accept
- Example
  - $L(M_1)$: Contains substring 01
  - $L(M_2)$: Contains an odd number of ones
  - $L(M_3)$: Contains 01 or has an odd number of 1s

Closure under Union

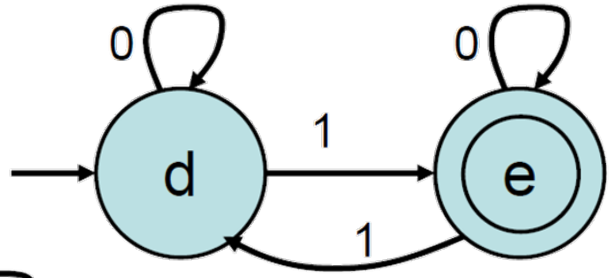
- Run both "in parallel"
- Symbols combine to become new states
  - $\Sigma_1 = \{a,b,c\}$
  - $\Sigma_2 = \{d,e\}$
  - $\Sigma_3 = \{ad,ae,bd,be,cd,ce\}$
- New states = accept if ordered pair contains old accepting state

- Example:

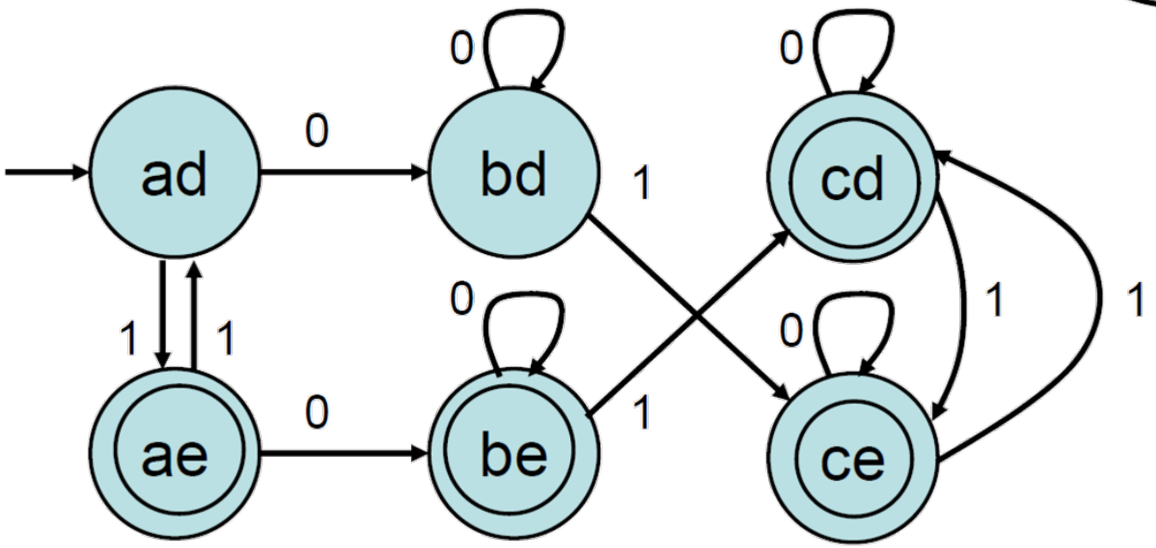
M_1: Substring 01



M_2: Odd number of 1s



M_3: 1



Closure under Union

- New Formal Definition
  - $M_1 = (Q_1,\Sigma_1,\delta_1,q_{01},F_1)$
  - $M_2 = (Q_2,\Sigma_2,\delta_2,q_{02},F_2)$

- Define $M_3 = (Q_3,\Sigma_3,\delta_3,q_{03},F_3)$, where
  - $Q_3 = Q_1 \times Q_2$
    - Cartesian Product, $\{ (q_1,q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \}$
  - $\Sigma_3 = \{0,1\}$
  - $\delta_3((q_1,q_2),a) = (\delta_1(q_1,a), \delta_2(q_2,a))$
  - $q_{03} = (q_{01},q_{02})$
  - $F_3 = \{ (q_1,q_2) \mid q_1 \in F_1 \text{ \underline{or} } q_2 \in F_2 \}$

Closure under Set Difference

- Theorem 4
  - FA-Recognizable languages are closed under set difference

- Proof
  - Similar proof to those for union and intersection
    - Accept if L_1 accepts <u>and</u> L_2 does not

  - Alternatively
    - Since $L_1 - L_2$ is the same as $L_1 \cap (L_2)^c$, just apply Theorems 1 and 2

Closure under Concatenation

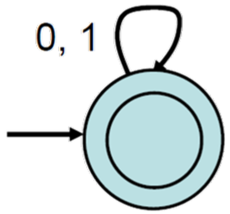
- Theorem 5: FA-Recognizable Languages are Closed under concatenation

- Proof
  - Start with FAs M_1 and M_2 for the same alphabet Σ
  - Get another FA, M_3, with
    - $L(M_3) = L(M_1) \circ L(M_2) = \{ x_1 x_2 \mid x_1 \in L(M_1) \text{ and } x_2 \in L(M_2) \}$

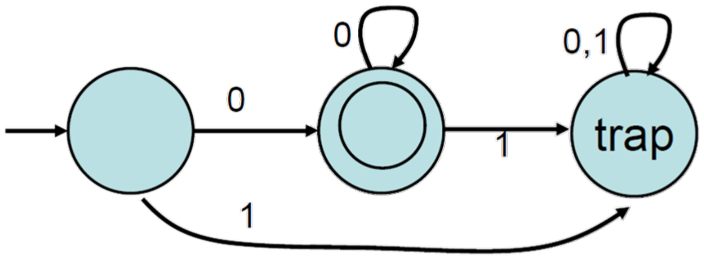
- Reasoning
  - Attach accepting states of M1 somehow to the start state of M_2
  - Don't know when string is done with M_1 portion of M_3
    - Careful as string may go through accepting states of M_1 several times

Closure under Concatenation

- Example
  - $\Sigma = \{0,1\}$, $L_1 = \Sigma^*$, $L_2 = \{0\}\{0\}^*$ (just zeros, at least one)
  - L_1L_2 = Strings that end with a block of at least one 0
    - M_1:



M_2:



- How to combine?
  - Need to "guess" when to shift to M_2
  - Leads to our next model, Nondeterministic Finite Automata
    - FAs that can guess
- Closure under star operation is an extension of this.