Nonregular Languages

Summary

- Nonregular Languages
- Prove that certain languages cannot be recognized by any finite automaton
- Pumping Lemma

Regular vs Nonregular Languages

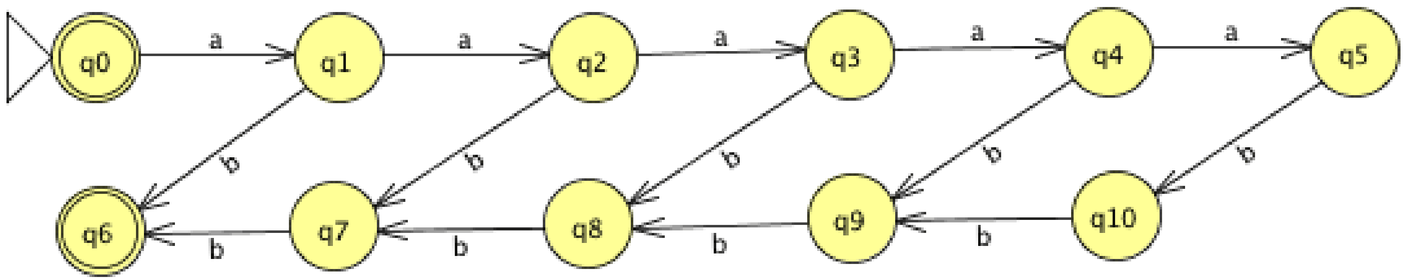
- **Regular languages**
  - Correspond to problems that can be solved with finite memory
    - i.e. finite states
- **Nonregular languages**
  - Correspond to problems that cannot be solved with finite memory
  - May need to remember one of infinitely many symbols
  - Requires infinite memory

Example of a Nonregular Language

- $L = \{ a^n b^n \mid n \geq 0 \}$
  - Because of n, we need the same number of a's and b's
    - {ε, ab, aabb, aaabbb, aaaabbbb, … }

  - If a^n and a^m ($n \neq m$) end up in the same state, $a^n b^n$ and $a^m b^n$ end up in the same state
    - DFA will either accept a string not in the language ($a^m b^n$) or reject a string in the language ($a^n b^n$)
    - This means for every n, we need a separate state

  - n is not limited, machine must track unlimited number possible states
    - Finite automata have a finite number of states and can not recognize this language
      - Nonregular Language



Must Prove Infinite Memory is Required

- Languages may not require infinite memory even though it seems so

- Example
  - D = { w|w has an equal number of occurrences of 01 and 10 as substrings }
    - Seems to require the need for counting occurrences
    - However, can be described by the following regular expression
      - $(1^{+}0^{*}1^{+})^{*} \cup (0^{+}1^{*}0^{+})^{*}$
      - D is a regular language

- Easy to prove a language is regular
  - Create a finite automata that recognizes it
  - Create a regular expression to describe language

- Harder to prove a language is nonregular
  - Must use other proof methods such as contradictions.

Methods to Prove Irregularity

- Proof by contraction of a property that is required by a regular language

- 3 properties are required for a regular language
  1. Closure of language under regular operations (i.e. union, intersection, complement, star…)

  2. Pumping Lemma

  3. Myhill-Nerode Theorem (won't be on exams)
     1. Strings x and y are **distinguishable** by language L if some string z exists whereby exactly one of the strings xz or yz belongs to L
     2. Let X be a set of strings where every 2 district strings are distinguishable.
     3. Let the index of L be the maximum number of elements in X
     4. The theorem states that L is regular iff it has a finite index
        1. In addition, the index is equal to the size of the smallest DFA that recognizes it.

Pumping Lemma

- **Pumping Lemma**
  - If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, s = xyz, satisfying the following conditions:
    - For each $i \geq 0, \quad xy^iz \in A$
    - $|y| > 0$, and
    - $|xy| \leq p$

- p is usually chosen as the number of states in a DFA.
  - If there are no strings in A that are at least length p, then pumping lemma <u>holds</u>.

- Used to show the irregularity of a language
  - Regular languages always satisfy the pumping lemma
  - Opposite is <u>not</u> true
    - If pumping lemma holds, it does <u>not</u> mean the language is regular

Pumping Lemma Proof

- M is a DFA that recognizes language A.
  - Let p = |Q| (the number of states in M)
  - Any string at least of length p can be broken into xyz parts

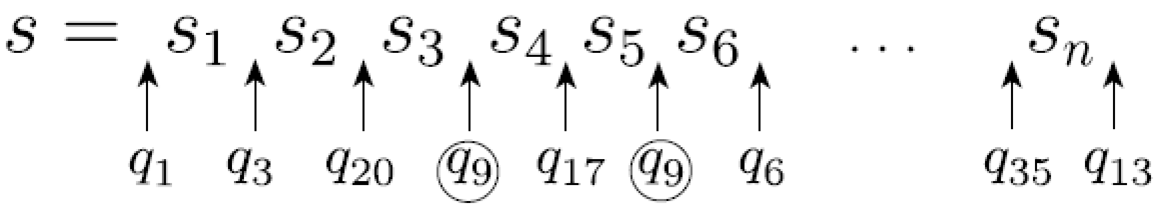
- Given string s of at least length p
  - If s is length n, it transitions into n+1 states
  - n+1 is greater than p
    - By the pigeonhole principle, some states are repeated

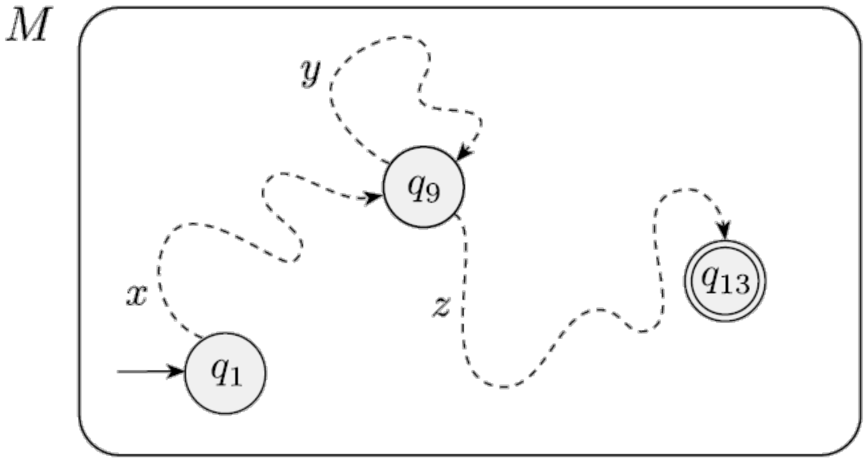
Pumping Lemma Proof

- If q_9 is the state that repeats, s can be divided into xyz with respect to q_9
  - x = substring before q_9, z = substring after q_9
  - y = substring between q_9 occurrences
    - **2nd condition** holds because $|y| \geq 1 > 0$

- This shows that the **1st condition** of the pumping lemma is satisfied
  - xy^iz is a string of A
    - No matter how many times we use y, it will be accepted because of z

- By the pigeonhole principle, a repeat must have happened by p+1
  - Since y is the repeatable portion,
    $|xy| \leq p$ (**3rd condition**)





Pumping Lemma

- All strings longer than the pumping length, p, can be "pumped"
  - Contains a section of the string that can be repeated any number of times to create new strings that are a part of the language
- All regular languages have the property stated by the pumping lemma
  - If the language does not have the property, it is nonregular
  - Can be used with proof by contradiction to show that a language is nonregular

Example 1

- Show that $L = \{0^n1^n | n \geq 0\}$ is non-regular using pumping lemma
- Suppose there is a DFA for L with p states
- Find a word w and pump to get a contradiction
- Choose $w = 0^p1^p$
  - Let w = xyz and pump to xyyz
  - Contradiction by the following 3 cases
    1. y is all zeros: xyyz has more zeros than ones and does not satisfy L's conditions
    2. y is all ones: xyyz has more ones
    3. y is a mix of ones and zeros: xyyz contains a 1 before a 0 which makes the string not member of L

Example 2

- Show that L = $\{ss|s \in \{0,1\}^*\}$ is non-regular using pumping lemma

- Choose w = $0^p 1 0^p 1$, p = number of states
  - Because of condition 3 of the pumping lemma, $|xy| \leq p$
    - xy is all zeros
    - Pumping y makes the string uneven dissatisfying the ss condition of L
  - e.g. w = 00010001, x = 0, y= 00, z = 10001
    - xyyz = 0000010001 ≠ ss

Example 3: Palindromes

- Show that L = $\{w \in \{0,1\}^* \mid w = w^{reverse}\}$ is non-regular using pumping lemma
- Choose w = $0^p 1 0^p$
  - Since $|xy| \leq p$, xy is all zeros
  - Since $|y| > 0$, y has at least 1 zero
  - xyyz is not a Palindrome
    - e.g. w = 0001000, x =00, y= 0, z = 1000, xyyz = 00001000

Example 4

- Show that L = {w∈{0,1}*|w contains the same number of zeros and ones} is non-regular using pumping lemma

- Choose $w = 0^p1^p$
  - Since $|xy| \leq p$, xy is all zeros
  - Since $|y| > 0$, y contains atleast 1 zero
  - xyyz does not contain an equal number of ones and zeros
  - e.g. w = 000111, x = 0, y = 00, z = 111, xyyz = 00000111

Example 5

- Show that $L = \{1^n | n \text{ is a prime number}\}$ is non-regular using pumping lemma
- Choose $w = 1^n$, with $n \geq p$
- $w = 1^n = xyz = 1^a 1^b 1^c$
- Pumping y does not guarantee that xy^iz will have a prime number of ones
  - Contradiction

Example 6: pump down

- Show that L = $\{0^i1^j | i>j\}$ is non-regular by pumping lemma

- Can't pump up since $i>j$

- Choose $w = 0^{p+1} 1^p$
  - Since $|xy|<p$, xy is all zeros
  - Since $|y|>0$, y has atleast one zero
  - Removing y will mean $i \leq j$, contradiction
  - e.g w = 0000111, x = 000, y = 0, z = 111
    - xyyz =00000111 is in L but
    - xz = 000111 isn't

Answering Questions about FAs

- We can ask general questions about DFAs, NFAs, and regular expressions and try to answer them algorithmically, that is, by procedures that could be programmed in some ordinary programming language
- Represent the DFAs, etc., by strings in some standard way, e.g., tuples with some encoding of a transition table
- Sample questions:
  - Acceptance: Does a given DFA M accept a given input string w?
  - Non-emptiness: Does DFA M accept any strings at all?
  - Totality: Does M accept all strings?
  - Nonempty Intersection: Do $L(M_1)$ and $L(M_2)$ have any string in common?
  - Subset: Is $L(M_1)$ a subset of $L(M_2)$?
  - Equivalence: Is $L(M_1)$ equal $L(M_2)$?
  - Finiteness: Is L(M) a finite set?
  - Optimality: Does M have the smallest number of states for a DFA that recognizes L(M)?